On achievable accuracy for range finder localization

How precise can a localization method be?

Given a sensor, there is an hard limit: the Cramér–Rao bound.

Summary:

1. Definition of the Cramér–Rao bound.
2. Application to range-finder localization.
3. A model for unstructured environments.
4. Experiments: predicting the error of the ICP.

The Cramér–Rao Bound

- A localization algorithm A is an estimator of $\boldsymbol{x}$ given function A from the sensor data $\boldsymbol{z}$: $\hat{\boldsymbol{x}} = A(\boldsymbol{z})$
- The estimator is unbiased if $E\{A(\boldsymbol{z})\} = \boldsymbol{x}$.

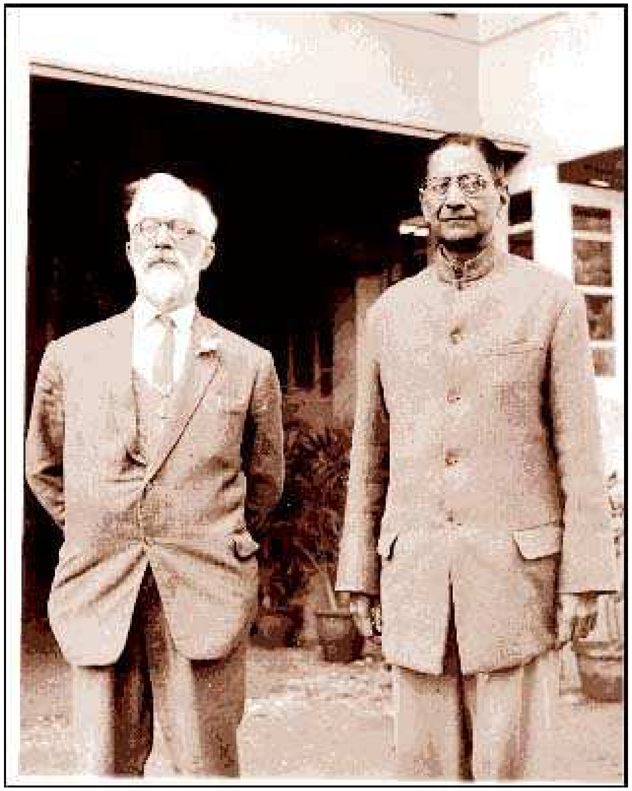
Cramér–Rao inequality: *For any unbiased estimator $\hat{\boldsymbol{x}}$,*

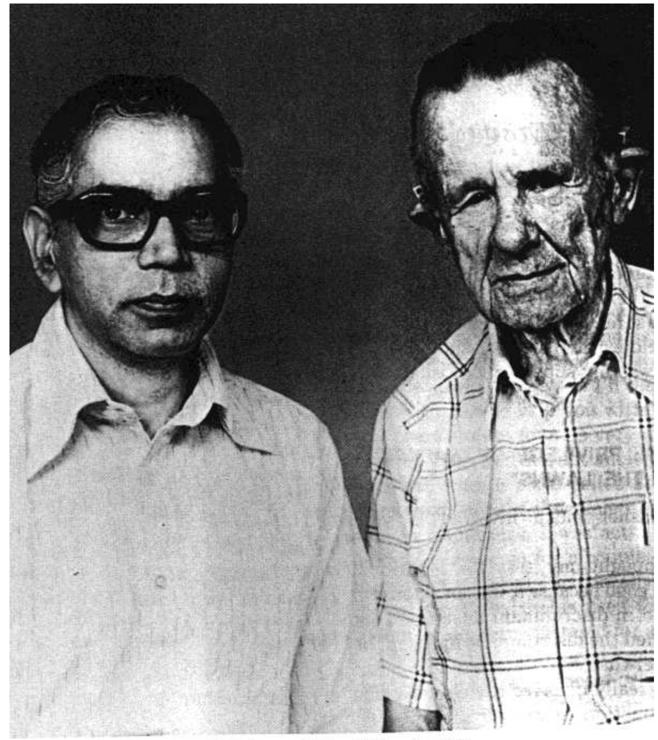
$$Cov(\hat{\boldsymbol{x}}) \geq (\boldsymbol{\mathcal{I}}(\boldsymbol{x}))^{-1}$$

The $n \times n$ symmetric matrix $\boldsymbol{\mathcal{I}}(\boldsymbol{x})$, called Fisher's information matrix, *is defined as*

$$\boldsymbol{\mathcal{I}}(\boldsymbol{x}) = E_z \left\{ \frac{\partial \log p(\boldsymbol{z}, \boldsymbol{x})}{\partial \boldsymbol{x}^T} \frac{\partial \log p(\boldsymbol{z}, \boldsymbol{x})}{\partial \boldsymbol{x}^T}^T \right\}$$

Fisher, Mahalanobis, Rao, Cramér





C. R. Rao and Harald Cramér, 1978

- Sir Ronald Ayimer Fisher (1890-1962), England
- Prasanta Chandra Mahalanobis (1893-1972), India
- Harald Cramér (1893-1985), Sweden
- Calyampudi Radhakrishna Rao (1920-), India

The Cramér–Rao lower bound

- The CRB is valid only for unbiased estimators.
  - Consider the localization algorithm

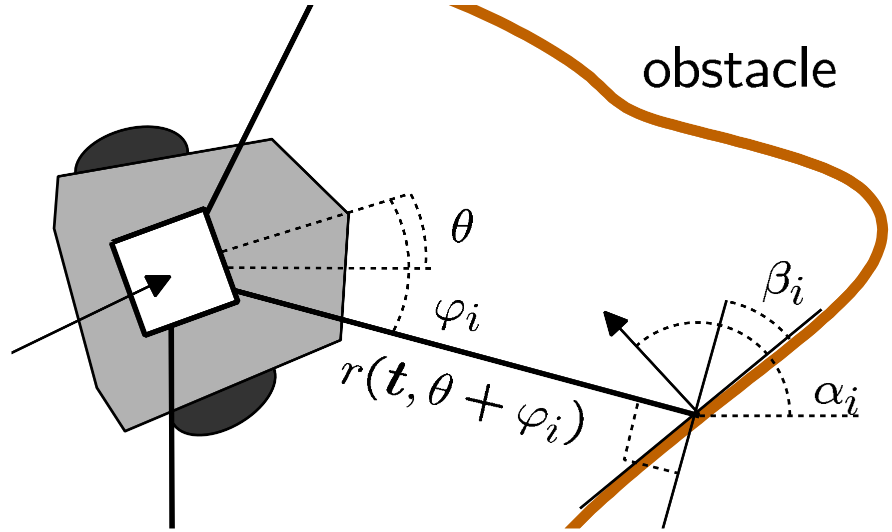
    $$A(\boldsymbol{z}) = (42, 42, 42^\circ)$$

    It has a covariance of 0.
  - Biased estimators might have a lower Mean Square Error:

    $$\text{MSE}_A(x) = \text{var}_A(x) + \textsf{bias}^2_A(x)$$

- The CRB It is not tight for non-Gaussian problems.
  - Localization is not a problematic problem.
  - Non-linear bounds are very very hard to derive.

CRB for localization

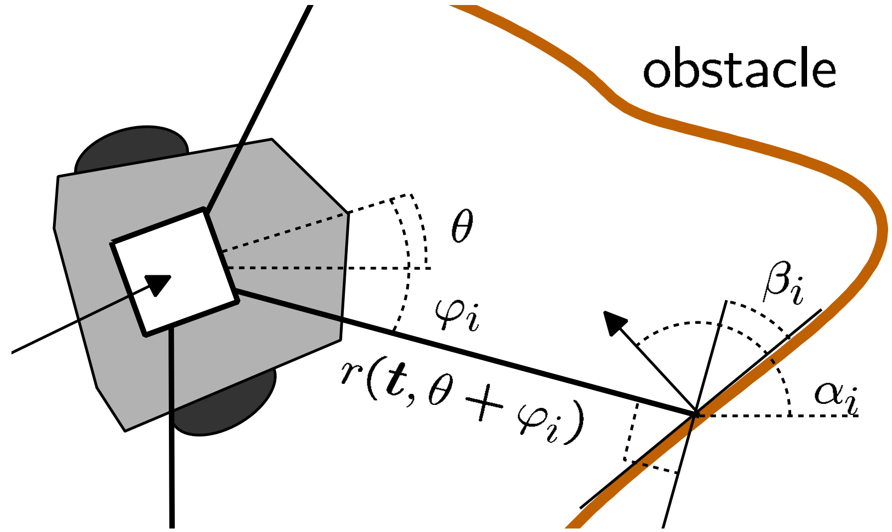


The likelihood function is

$$p(\boldsymbol{z}|\boldsymbol{x}) = \prod_i \mathcal{N}(\tilde{\rho}_i - r(\boldsymbol{t}, \theta + \varphi_i), \sigma^2)$$

$r(\boldsymbol{p}, \psi)$ is the "raytracing function".

CRB for localization

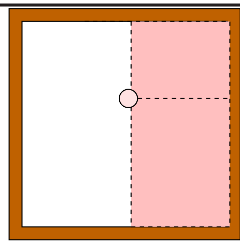
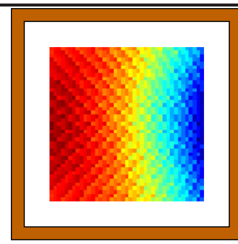
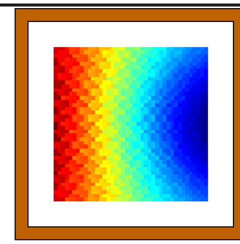
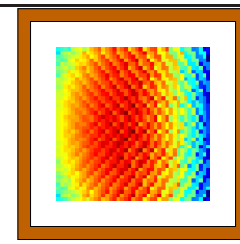
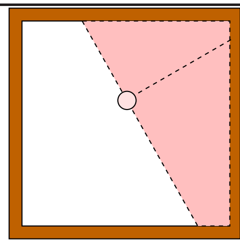
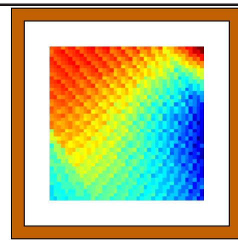
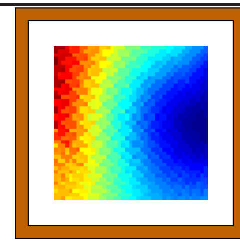
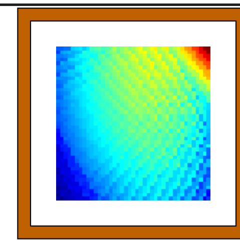
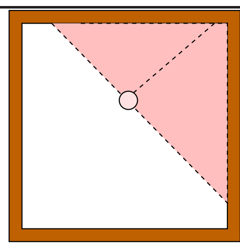
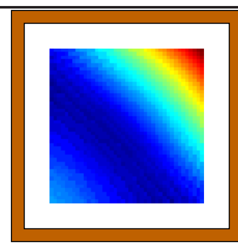
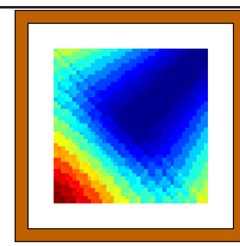
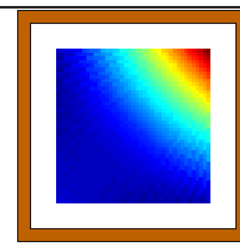


Fisher's information matrix is

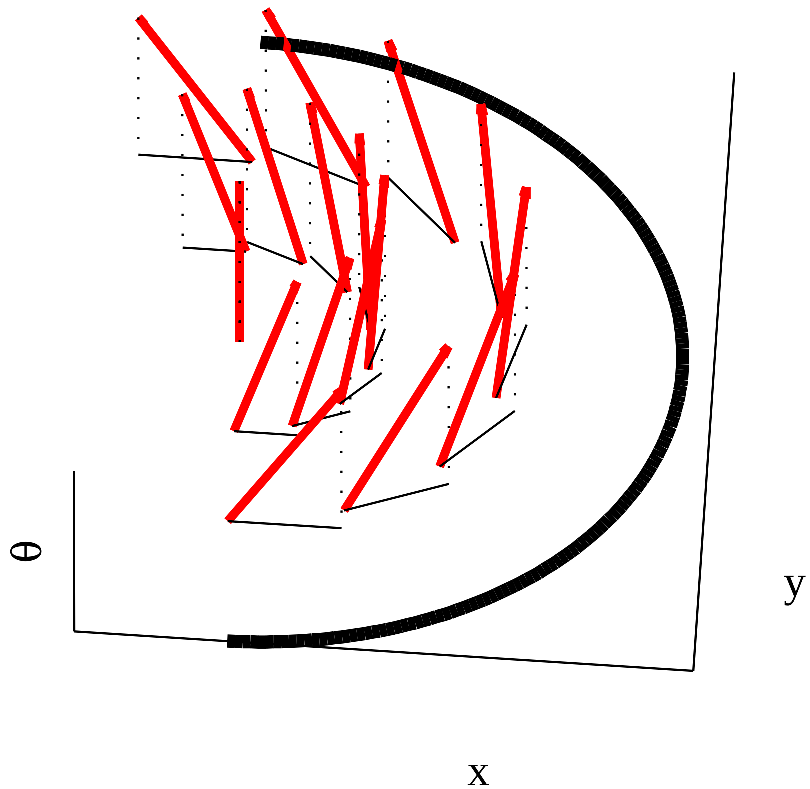
$$\boldsymbol{\mathcal{I}}(\boldsymbol{x}) = \frac{1}{\sigma^2} \sum_i \begin{bmatrix} \dfrac{\boldsymbol{v}(\alpha_i)\boldsymbol{v}(\alpha_i)^T}{\cos^2 \beta_i} & r_i \dfrac{\tan \beta_i}{\cos \beta_i} \boldsymbol{v}(\alpha_i) \\ * & r_i^2 \tan^2 \beta_i \end{bmatrix}$$

α is the surface direction, $\beta \triangleq \alpha_i - (\theta + \varphi_i)$

Results in a square environment

Orientation	$\det(\mathrm{cov}(\hat{\boldsymbol{x}}))$	$\det(\mathrm{cov}(\hat{\boldsymbol{t}}))$	$\mathrm{var}(\hat{\theta})$
			
			
			

Kernel of $\mathcal{I}(x)$

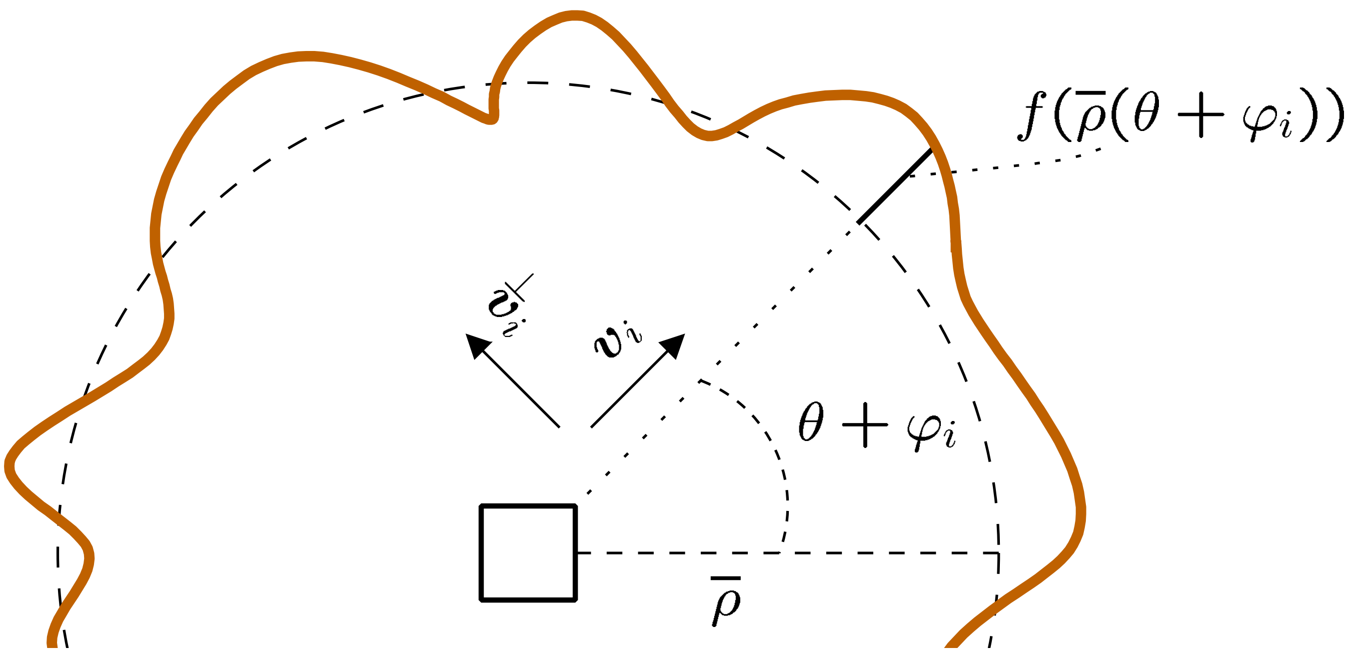


In under-constrained situations, the kernel of $\mathcal{I}(x)$ gives the direction of uncertainty.

A model for unstructured environments

Consider the environment (defined through the raytracing function)

$$r(\mathbf{0}, \phi) = \overline{\rho} + f(\phi \overline{\rho}) \qquad |f| \ll \overline{\rho}$$



A model for unstructured environments

For this model, the following approximation holds:

$$\mathrm{var}(\hat{\theta}) \geq \frac{(\sigma/\overline{\rho})^2}{n}\frac{1}{\mathcal{C}}$$

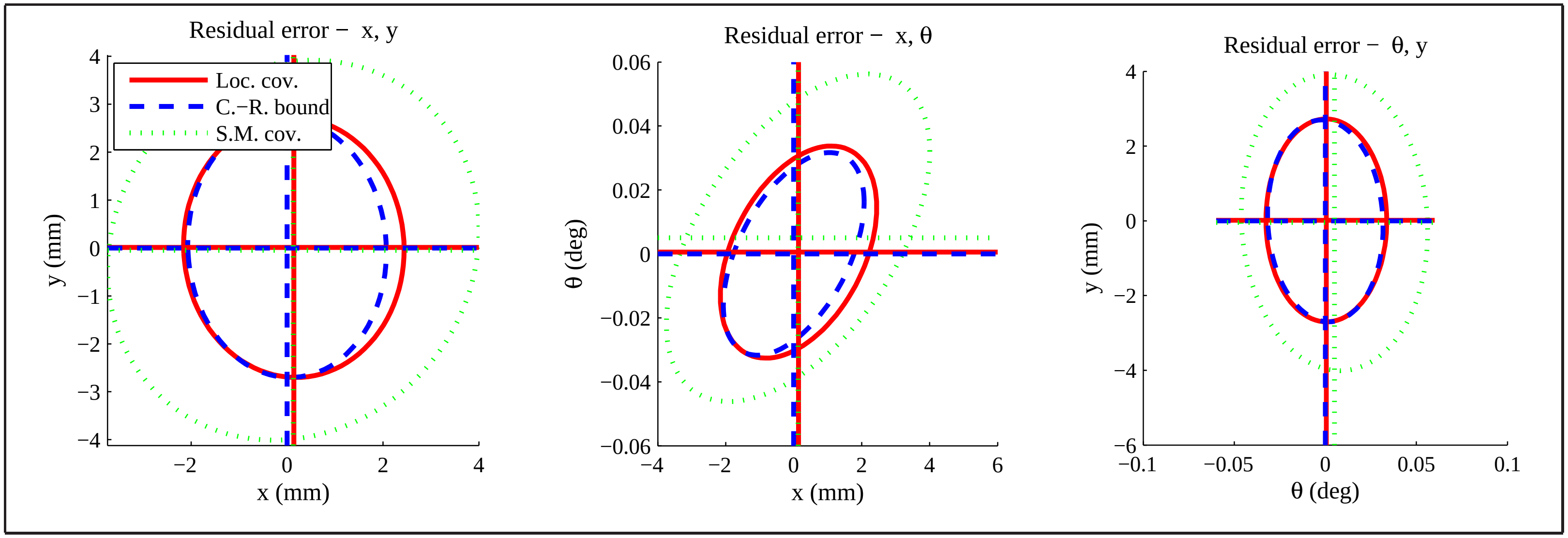
$$\lambda_{\min}\mathrm{cov}\left(\hat{\boldsymbol{t}}\right) \geq 2\frac{\sigma^2}{n}\frac{1}{1+\mathcal{C}}$$

where $\mathcal{C} \triangleq E_f\{f'^2\}$.

- Localization is easier with features (high values of $\mathcal{C}$).
- For a circle: $\mathcal{C} = 0$. The uncertainty on θ is infinite, while the uncertainty for $\boldsymbol{t}$ is still finite.
- The accuracy for θ depends on the "normalized" noise $\sigma/\overline{\rho}$ (invariance to scale)
- $\mathrm{cov}\left(\hat{\boldsymbol{t}}\right)$ does not depend on $\overline{\rho}$.

Experiments

The CRB predicts very well the uncertainty of the ICP in localization, while it is a weak bound for scan matching.



Asymmetric situation: square environment ($5m$ side),
FOV $= 180°, \boldsymbol{x} = (-2m, 2m, 30°)$

Future work

- Localization is a finite-dimensional problem, while SLAM is infinite-dimensional.
- First explore *mapping:* the key is to choose a good representation for the map.
  - Polygonal environment: very easy to obtain the CRB for the map.
  - Occupancy grids: inference is tricky.
  - Splines.
  - Gaussian processes.

Predicting localization accuracy in a filter

Update equation of a Bayesian filter:

$$p(\boldsymbol{x}_t|t) \propto \underbrace{p(\boldsymbol{z}_t|\boldsymbol{x}_t)}_{\text{likelihood}} \underbrace{\int p(\boldsymbol{x}_t|\boldsymbol{x}_{t-1}, \boldsymbol{u}_t) p(\boldsymbol{x}_{t-1}|t-1) d\boldsymbol{x}_{t-1}}_{p(\boldsymbol{x}_t|t-1)}$$

If everything is Gaussian:

$$\boldsymbol{\Sigma}_{\boldsymbol{x}_t|t} = \left(\boldsymbol{\Sigma}^{-1}_{\boldsymbol{z}_t|\boldsymbol{x}_t} + \left(\boldsymbol{\Sigma}_{\boldsymbol{x}_{t-1}|t-1} + \boldsymbol{\Sigma}_{\boldsymbol{u}} \right)^{-1} \right)^{-1}$$

If your localization algorithm is efficient:

$$\boldsymbol{\Sigma}_{\boldsymbol{x}_t|t} \simeq \left(\boldsymbol{\mathcal{I}}(\boldsymbol{x}_t) + \left(\boldsymbol{\Sigma}_{t-1} + \boldsymbol{\Sigma}_u \right)^{-1} \right)^{-1}$$