Università di Roma **La Sapienza**
Dipartimento di Informatica e Sistemistica

An accurate closed-form estimate of ICP's covariance

Andrea Censi

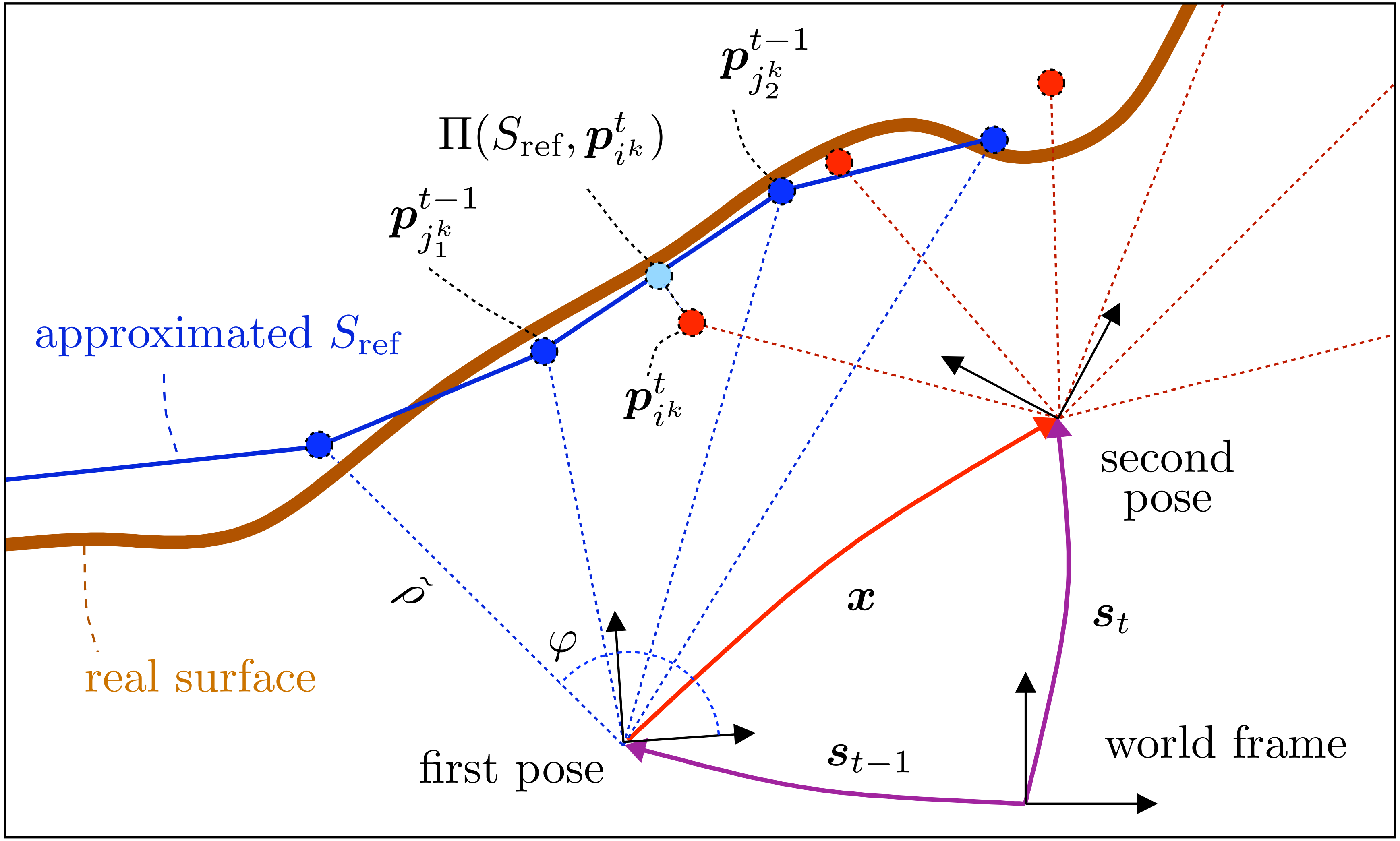
- Based on the analysis of the error function.
- Advantages over previous approaches:
  - It does not assume independent point correspondences.
  - Measurements can be correlated.
  - Both **accurate** and **fast** (closed form).

The vanilla ICP algorithm

- Input:
  - a reference surface S_{ref} (created from the first scan $\boldsymbol{z}_1$)
  - a second sensor scan $\boldsymbol{z}_2$
  - a starting guess $\boldsymbol{x}_0$
- Repeat until convergence:
  1. compute a set of correspondences
  2. define an error function $J(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{x})$
  3. adjust roto-translation $\boldsymbol{x}$ to minimize J
- Can be used for scan matching and localization.
- Many flavours available...

The vanilla ICP algorithm

- Reference surface is created with a polyline.
- Three points involved for each correspondence.



Why do I get a wrong solution?

- There is no *right* or *wrong* in statistics.
- Sometimes there just is **not enough information** (under-constrained situations, such as a corridor).

under-constrained situations can be detected using Fisher's matrix

"On achievable accuracy for range-finder localization"

today in the 'miscellaneous' ***session FrC9 at 14:45***

- Sometimes ICP goes crazy due to a **bad initial guess**.

  this is hard to model; we assume that it converges to the right basin.

- And then, there is regular **sensor noise**.
- Why is it hard to estimate the ICP covariance?
  - The correspondence/minimize/repeat loop is hard to analyze.

- The map, created from the first scan, is noisy.

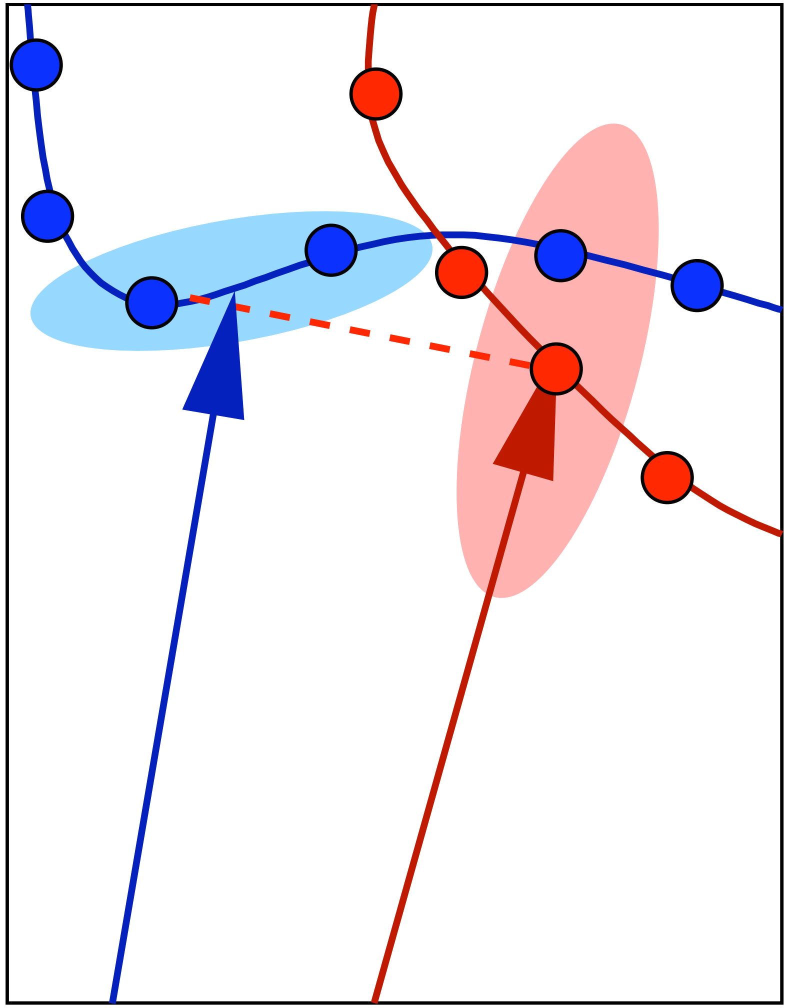
Related work - the naive method

- Associate a covariance matrix to each sensor point.
- Assume each correspondence is an <u>independent</u> observation.

$$\mathrm{cov}(\hat{\boldsymbol{x}})^{-1} = \sum_k (P_k)^{-1}$$

- Limitations:
  - In practice, very optimistic.
  - Correspondences are not independent.
  - Environment structure is important

    Pfister *et al.* (2002); Montesano *et al.* (2005)



Related work - The "brute force method"

- Monte Carlo approximation to the real covariance. Bengtsson (2006)
- Algorithm:
  1. Approximate a map $\tilde{S_{\text{ref}}}$ using the first scan.
  2. Repeat multiple times (> 50):
     (a) Choose a random displacement $\boldsymbol{x}_k$.
     (b) Simulate a sensor scan from $\tilde{S_{\text{ref}}}$.
     (c) Run ICP and compute the error $\hat{\boldsymbol{x}} - \boldsymbol{x}_k$.
  3. Compute the covariance of the errors.
- Limitations:
  - Computationally expensive.
  - One must simulate using an **imperfect** map.

Related work - the "Hessian" method

- If the problem was linear, and the error function was quadratic:

$$\boldsymbol{z} = M\boldsymbol{x} + \sigma^2\epsilon \qquad \Rightarrow \qquad \mathrm{cov}(\hat{\boldsymbol{x}}) = \sigma^2 \left(\frac{\partial^2 J}{\partial \boldsymbol{x}^2}\right)^{-1}$$

- Idea: pretend the problem is linear: Bengtsson (2006)

$$\mathrm{cov}(\hat{\boldsymbol{x}}) = \underbrace{2\,\frac{J(\boldsymbol{z}, \hat{\boldsymbol{x}})}{K-3}}_{\text{approximation to } \sigma^2} \left(\frac{\partial^2 J}{\partial \boldsymbol{x}^2}\right)^{-1}$$

- Get a robust Hessian by sampling. Biber and Strasser (2003), etc.
- Limitations:
  - In practice, sometimes very pessimistic.
  - Not sound: the Hessian is just part of the solution.

The covariance of a minimization algorithm

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}(\boldsymbol{z}) = \arg\min J(\boldsymbol{z}, \boldsymbol{x})$$

- Because $\hat{\boldsymbol{x}}$ is a stationary point of the gradient: $\nabla J(\boldsymbol{z}, \hat{\boldsymbol{x}}) = 0$, the implicit function theorem provides the $\boldsymbol{z} \to \hat{\boldsymbol{x}}$ Jacobian.

$$\underbrace{\mathrm{cov}(\hat{\boldsymbol{x}})}_{\substack{\text{solution}\\ \text{covariance}}} = \underbrace{\left(\frac{\partial^2 J}{\partial \boldsymbol{x}^2}\right)^{-1} \frac{\partial^2 J}{\partial \boldsymbol{x} \partial \boldsymbol{z}}}_{\frac{\partial \hat{\boldsymbol{x}}}{\partial \boldsymbol{z}} \text{ Jacobian}} \underbrace{\mathrm{cov}(\boldsymbol{z})}_{\substack{\text{input}\\ \text{covariance}}} \frac{\partial^2 J}{\partial \boldsymbol{x} \partial \boldsymbol{z}}^T \left(\frac{\partial^2 J}{\partial \boldsymbol{x}^2}\right)^{-1}$$

- All is evaluated at the minimum $\hat{\boldsymbol{x}}$.
- Contains the Hessian **and** the mixed derivative $\partial^2 J / \partial \boldsymbol{x} \partial \boldsymbol{z}$: how the shape changes with respect to the measurements.
- Reduces to a familiar formula if J is quadratic (try it).

Application to ICP

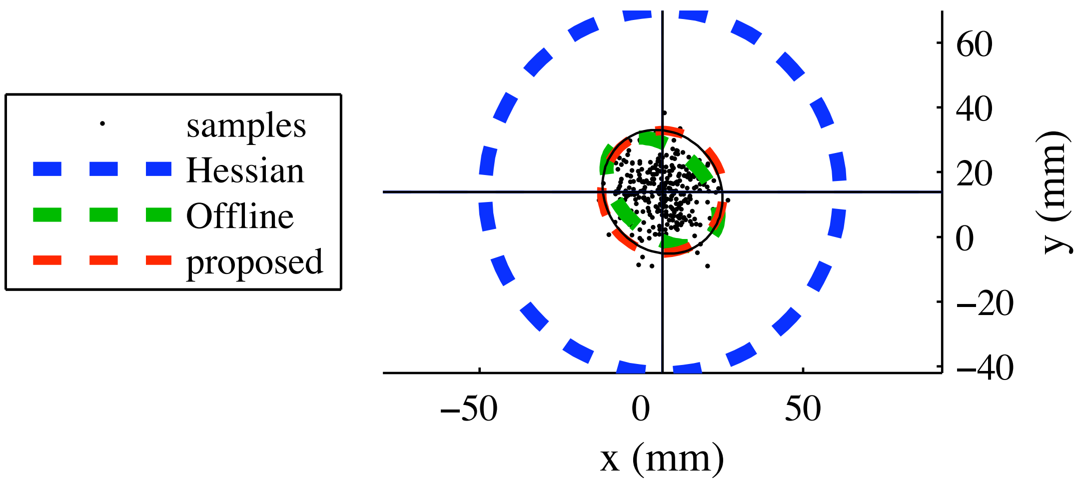
- How to implement this:
  1. run ICP to get $\hat{\boldsymbol{x}}$
  2. evaluate the derivatives at $\hat{\boldsymbol{x}}$
     - closed form is possible (lengthy but simple formulas)
- Note that:
  - If the same measurements contributes to two different correspondences, that is taken into account in $\frac{\partial^2 J}{\partial \boldsymbol{x} \partial \boldsymbol{z}}$.

$$\begin{matrix}\text{Error}\\\text{function}\end{matrix} = \sum_{\text{correspondences}} \text{term involving 3 measurements}$$

  - The measurement matrix $\text{cov}(\boldsymbol{z})$ can be a full matrix.

Experiments - scan matching

- the Hessian method is very pessimist.
- the proposed method is very accurate
  - better than the Offline method (!)



Scan matching errors (*mm*,*mm*,°)			
	$\sigma(x)$	$\sigma(y)$	$\sigma(\theta)$
true	7.6	7.8	0.058
Hessian	20.0	20.3	0.171
Offline	7.0	6.8	0.086
proposed	7.7	7.7	0.060

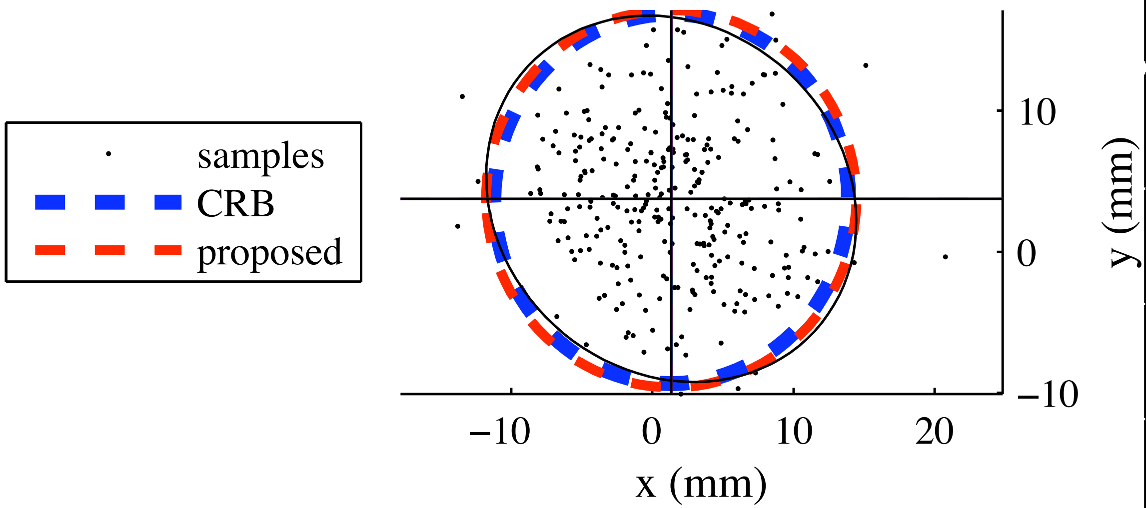
- An observability analysis is needed in under-constrained situations; proposed $\simeq$ Hessian, slightly optimistic.

Experiments - localization

- The method can also be used for localization
  - the map is assumed to be perfect.

$$\mathrm{cov}(\boldsymbol{z}) = \begin{pmatrix} \mathrm{cov}(\boldsymbol{z}_1) & 0 \\ 0 & \mathrm{cov}(\boldsymbol{z}_2) \end{pmatrix} \quad \text{becomes} \quad \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{cov}(\boldsymbol{z}_2) \end{pmatrix}$$

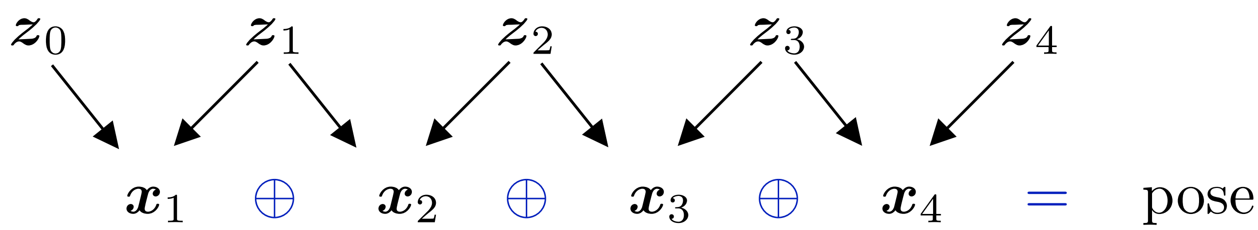
- ... one has the same results as the Cramér–Rao bound.



Localization errors (*mm*,*mm*,°)			
	$\sigma(x)$	$\sigma(y)$	$\sigma(\theta)$
true	5.3	5.3	0.039
CRB	5.1	5.3	0.037
proposed	5.4	5.4	0.042

Correlation among successive poses

- In scan matching, each sensor scan is used twice.



Hence, the estimated displacements $\boldsymbol{x}_k$ are not independent.

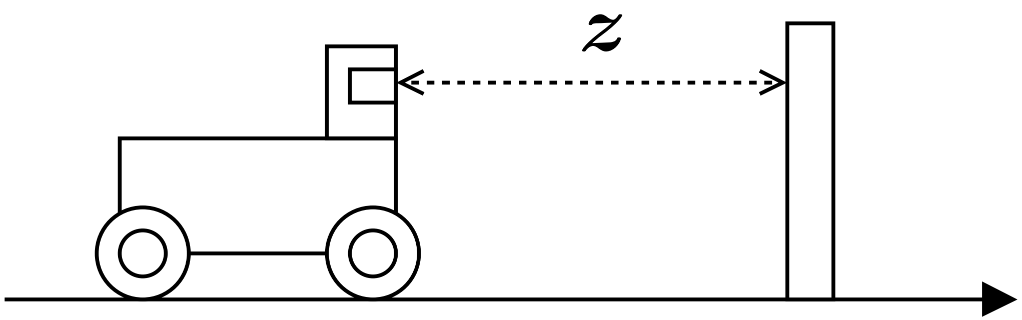
- If you just "sum" covariances, you would be pessimist, as scan matching errors tend to cancel out.

- Problem solved by Mourikis and Roumeliotis (2006): you just need the Jacobians $\frac{\partial \boldsymbol{x}_k}{\partial \boldsymbol{z}_k}$ and $\frac{\partial \boldsymbol{x}_k}{\partial \boldsymbol{z}_{k+1}}$ which we computed:

$$\frac{\partial \hat{\boldsymbol{x}}}{\partial \boldsymbol{z}} = \left[\begin{array}{cc} \frac{\partial \hat{\boldsymbol{x}}}{\partial \boldsymbol{z}_1} & \frac{\partial \hat{\boldsymbol{x}}}{\partial \boldsymbol{z}_2} \end{array} \right] = \left(\frac{\partial^2 J}{\partial \boldsymbol{x}^2} \right)^{-1} \frac{\partial^2 J}{\partial \boldsymbol{x} \partial \boldsymbol{z}}$$

Example: 1-dimensional scan matching

- In 1-dimensional scan matching, errors tend to cancel out.



$$\text{final pose} = \boldsymbol{x}_1 \oplus \boldsymbol{x}_2 \oplus \cdots \oplus \boldsymbol{x}_n \qquad \text{(sum of deltas)}$$
$$= (\boldsymbol{z}_1 - \boldsymbol{z}_0) + (\boldsymbol{z}_2 - \boldsymbol{z}_1) + \cdots + (\boldsymbol{z}_n - \boldsymbol{z}_{n-1}) = \boldsymbol{z}_n - \boldsymbol{z}_1$$

- The real final covariance is very small:

$$\text{var(final pose)} = 2\,\text{var}(\boldsymbol{z}) \qquad << 2n\,\text{var}(\boldsymbol{z})$$

- The more correlated the measurements are, the more you are being pessimist if you ignore the correlation.

Conclusions

- **A good trick to know**: The covariance of an minimization algorithm only depends on the error function.
- Advantages over previous methods for estimating ICP's covariance:
  - mathematically sound
  - accurate (also more than simulations-based methods)
  - fast: closed form
  - also solves the problem of correlated estimates
- For more on the observability analysis, Fisher's information matrix, Cramér–Rao bound, please see

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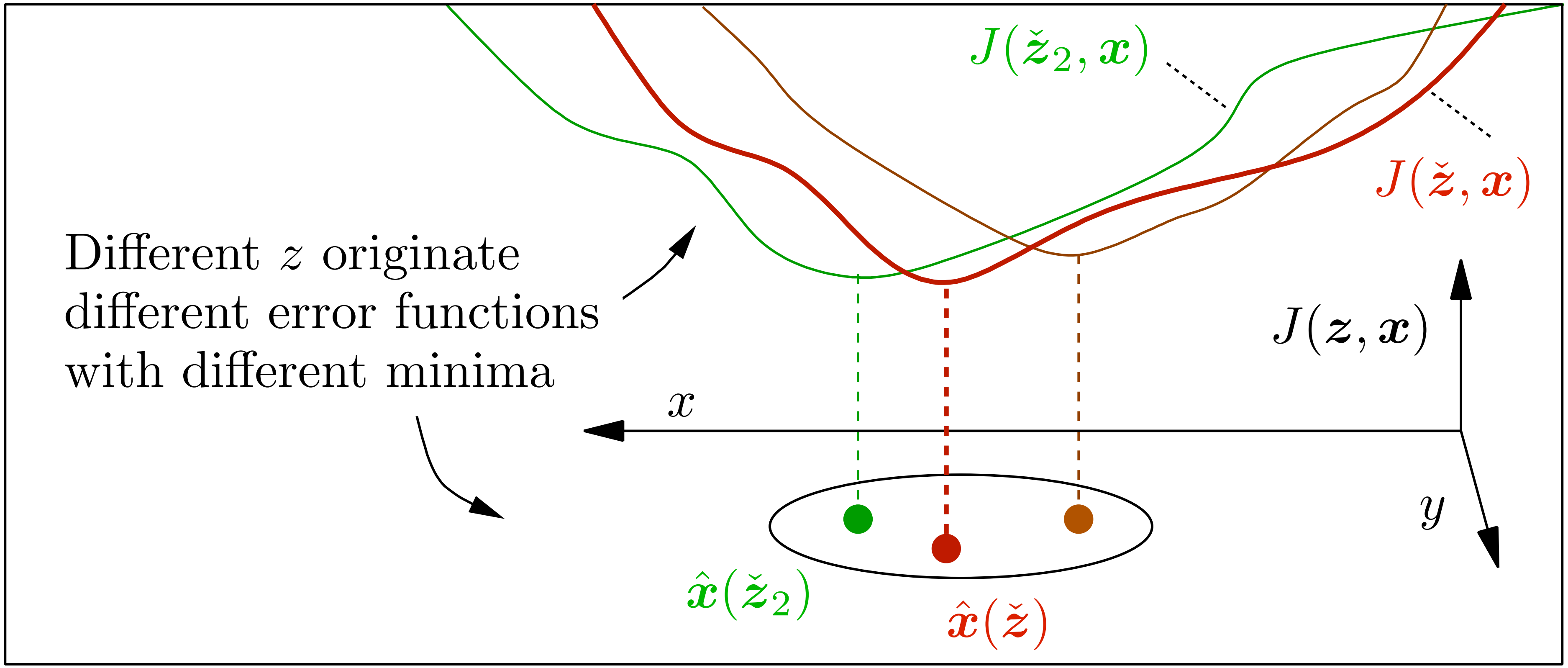
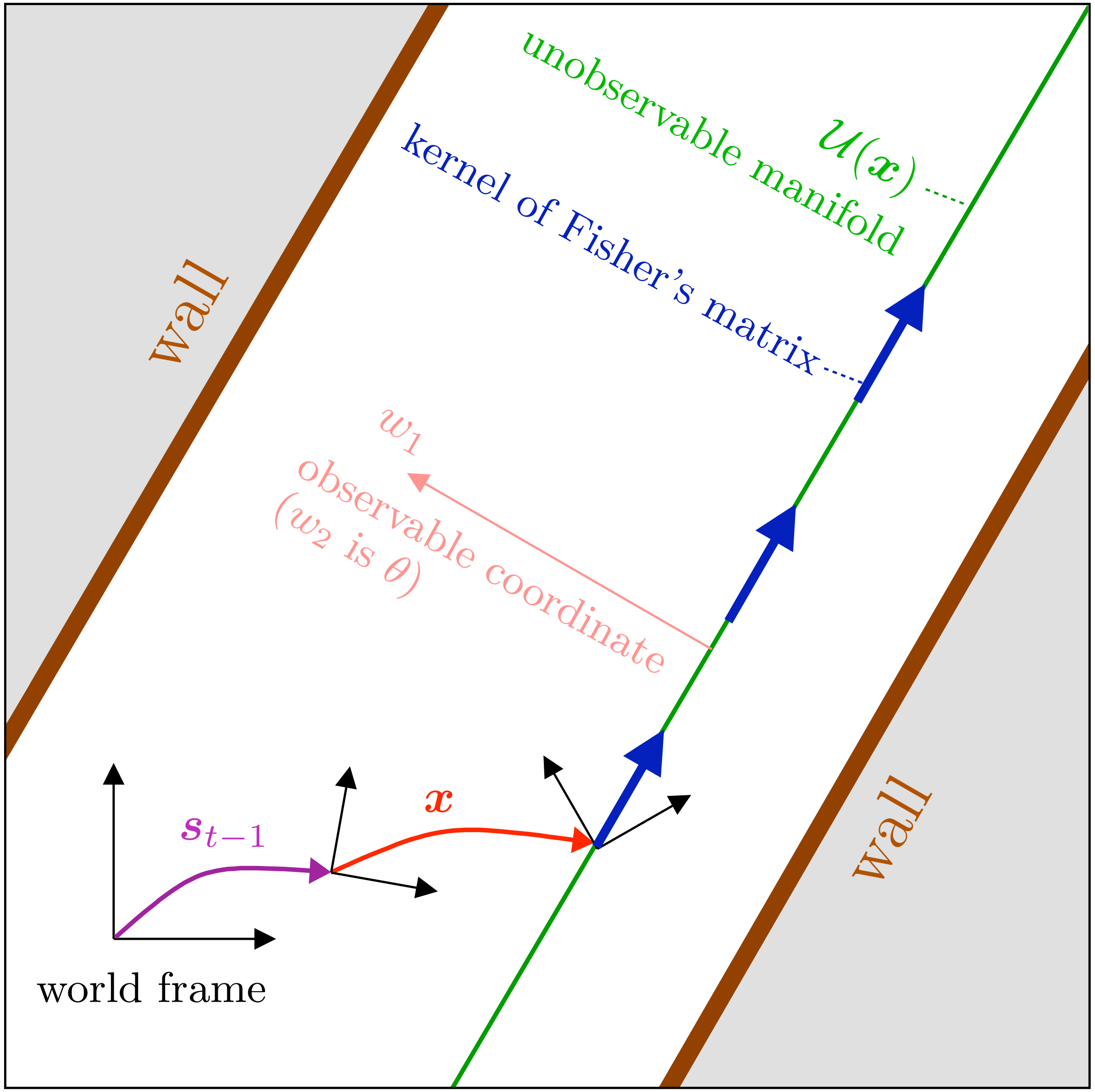
Backup / Change in the error function

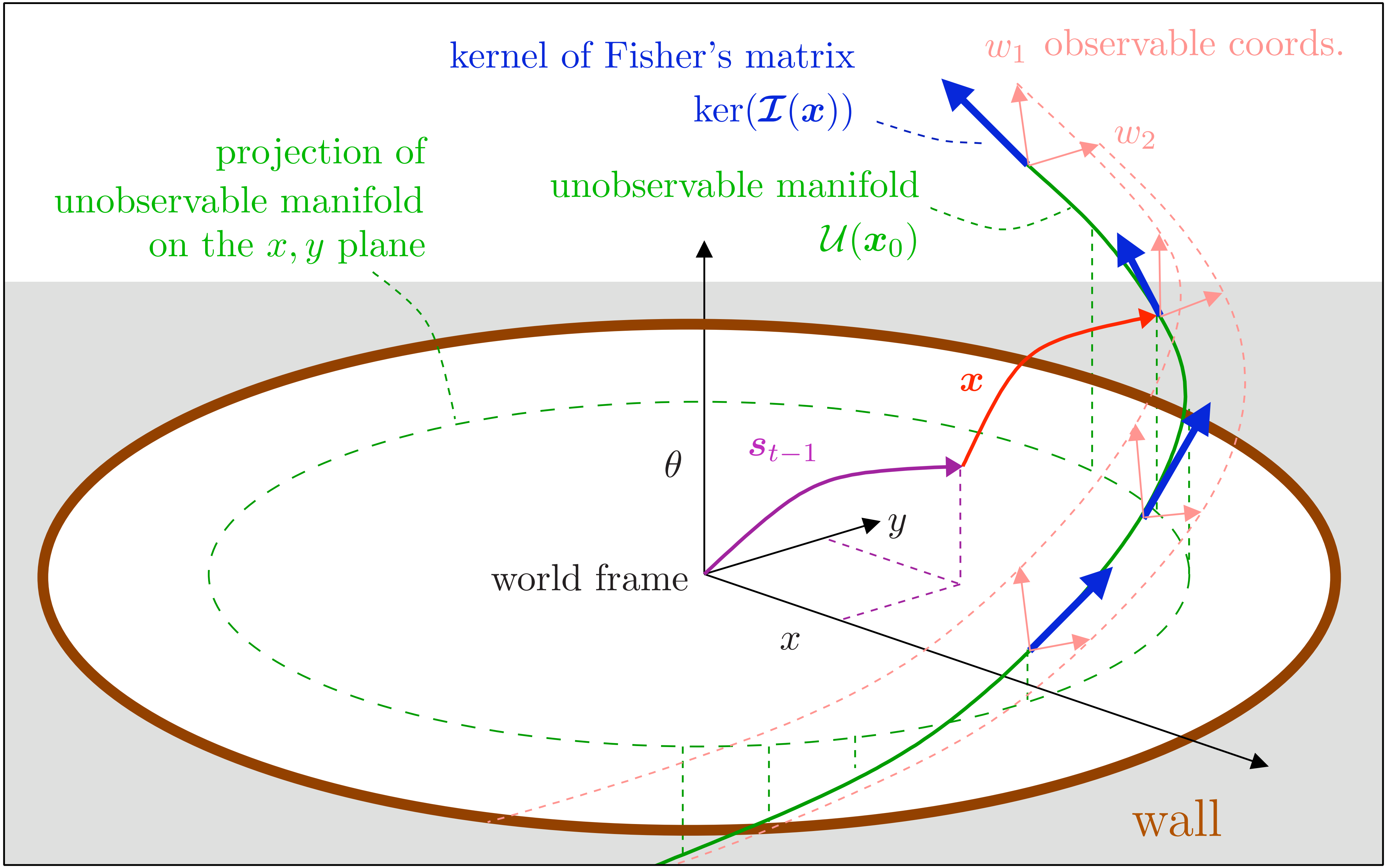
- The term $\dfrac{\partial^2 J}{\partial \boldsymbol{x} \partial \boldsymbol{z}} = \dfrac{\partial}{\partial \boldsymbol{z}} \nabla J$ accounts for the change in the shape of the error function.



Backup / Observable manifold – corridor



Backup / Observable manifold – circle



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