Real-valued average consensus over noisy quantized channels

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Consensus problems

- **Consensus**: reach the agreement of agent beliefs or agent states, respecting the given communication constraints.
- **Basic average consensus problem**: $x_i(k) \rightarrow \frac{1}{n} \sum x_i(0)$.

- Interesting to me because it is an example of **distributed computation** done by a network of simple units.
- Computation/control on distributed/noisy substrates will be an important topic:
  - neuronal networks (neuroscience)
  - noisy electronic components (precision vs. efficiency)
  - chemical reaction networks
Ideas from neuroscience

- The brain is the only instance of intelligence we know. We are very very far from understanding how it works.
- How about neurons?
  - Asynchronous distributed computation using spikes.
  - They are slow with respect to the dynamics they control (e.g. fruit fly).
  - They are noisy.
  - Lots of models (we don’t have a clue of what is important)
    - Simplest non-trivial: linear sum of inputs + noisy nonlinearity.

- Can a control theorist tell something interesting?

  Useless things to prove:
  - “stability”
  - “synchronization”

  Interesting things to prove:
  - computational properties
  - adaptation/learning

- Can a noisy spiking network solve the consensus problem?
Some related work

Real-valued consensus over quantized channels is a two-part strategy:
1. Communication strategy: decide the value $y_j(k) \in \mathbb{Z}$ to send.
2. Update strategy: update the node’s state $x_i(k)$ based on received $y_j(k)$

- [Aysal et al. ’07]: Given $P$ stochastic, let

$$y_j(k) = q^p(x_j(k)) \quad x_i(k) = \sum_j P_{i,j} y_j(k)$$

Uses “probabilistic quantization” $q^p(x) = \begin{cases} \lfloor x \rfloor & \text{with probability } x - \lfloor x \rfloor \\ \lceil x \rceil & \text{otherwise} \end{cases}$

Results: consensus is reached to a value $\tau \in \mathbb{Z}$; $\mathbb{E}\{\tau\} = \text{average}.$

- [Carli et al. ’08]: Given $P$ doubly stochastic, let

$$y_j(k) = \text{round}(x_j(k)) \quad x_i(k + 1) = x_i(k) - y_i(k) + \sum_j P_{i,j} y_j(k)$$

Results: the average is conserved; the consensus is not reached.
Model/approach

Update strategy: We adapt from [Olfati-Saber ’07]:

\[ x_i(k + 1) = x_i(k) + \frac{\eta}{\Delta} \sum_j a_{ij} (y_j(k) - x_i(k)) \]

- \( a_{ij} = a_{ji} \) is an element of the adjacency matrix; \( \Delta \) is the degree of the graph; \( \eta \in (0, 1) \) a parameter.

Communication strategy

- Assume \( y(k) = \psi(x(k)) \), with \( \psi \) arbitrary function:

\[ |\psi(x) - x| \leq \beta \]
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Communication strategy

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\[ y_j(k) = \psi(x_j(k) - c_j(k)) \]
\[ c_j(k+1) = c_j(k) + \underbrace{(y_j(k) - x_j(k))} \text{ transmission error} \]

Has the flavor of a self-inhibitory action potential.
Behavior of the drift

- Define the drift $d(k)$ as

\[
d(k) \triangleq \left| \frac{1}{n} \sum_i x_i(k) - \alpha \right|
\]

- $\alpha \triangleq \frac{1}{n} \sum_i x_i(0)$ is the goal state.

- **Proposition:** The drift is bounded:

\[
d(k) \leq \eta \beta
\]

- $\beta$ is the bound on the quantization error
- $\eta$ is the speed of the update strategy

- By choosing $\eta$, we can make the drift as small as desired.
Behavior of the disagreement error

- Take as an error measure the *average disagreement*:

\[
\varphi(k) \triangleq \left[ \frac{1}{n \Delta} \sum_{i,j} a_{ij} (x_i(k) - x_j(k))^2 \right]^{1/2}
\]

- \( \Delta \) is the degree of the graph \((n \Delta \simeq \text{number of edges})\)

- **Proposition:** Eventually, the disagreement is bounded by:

\[
|\varphi(k)| \leq \sqrt{6} \cdot \eta \beta \cdot \frac{\lambda_n \{\mathbf{L}\}}{\lambda_2 \{\mathbf{L}\}}
\]

- \( \lambda_2 \{\mathbf{L}\} \) is the second smallest eigenvalue, \(\neq 0\) if graph connected.
- \( \beta \) is the bound on the quantization error
- \( \eta \) is the speed of the update strategy

- By choosing \( \eta \), we can make the disagreement as small as desired.
## Comparison

<table>
<thead>
<tr>
<th>Method</th>
<th>Drift</th>
<th>Disagreement</th>
</tr>
</thead>
<tbody>
<tr>
<td>No quantization</td>
<td>$d(k) = 0$</td>
<td>$\varphi(k) \to 0$</td>
</tr>
<tr>
<td>Carli et al.</td>
<td>$d(k) = 0$</td>
<td>$\varphi(k) \to c &gt; 0$</td>
</tr>
<tr>
<td>Aysal et al.</td>
<td>$d(k) \neq 0$</td>
<td>$\varphi(k) \to 0$</td>
</tr>
<tr>
<td>Proposed strategy</td>
<td>$d(k) \leq \eta \beta$</td>
<td>$\lim_{k \to \infty} \varphi(k) \leq c \cdot \eta \beta \frac{\lambda_n{L}}{\lambda_2{L}}$</td>
</tr>
</tbody>
</table>

- Therefore, consensus can be reached with arbitrary precision.
- But small $\eta$ implies slow convergence.
Characterization of the bound

- For some graphs, $\lambda_n L / \lambda_2 L$ depends on the number of nodes $n$.
  - yet the performance appear to be largely independent of $n$

<table>
<thead>
<tr>
<th>graph</th>
<th>$\lambda_n L$</th>
<th>$\lambda_2 L$</th>
<th>$\lambda_n L / \lambda_2 L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>star</td>
<td>$n$</td>
<td>1</td>
<td>$n$</td>
</tr>
<tr>
<td>complete</td>
<td>$n$</td>
<td>$n$</td>
<td>1</td>
</tr>
<tr>
<td>ring</td>
<td>4</td>
<td>$2 - 2 \cos \left( \frac{2\pi}{n} \right)$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>path</td>
<td>$2 + 2 \cos \left( \frac{\pi}{n} \right)$</td>
<td>$2 - 2 \cos \left( \frac{\pi}{n} \right)$</td>
<td>$n^2$</td>
</tr>
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Examples

\( \psi = \text{round; ring graph with } n = 10 \text{ nodes.} \)

\( \eta = 0.1, \text{ overall behavior} \)

\begin{align*}
\eta = 0.1, \text{ overall behavior} \\
\text{States} \\
\text{Disagreement} \\
\text{Drift}
\end{align*}

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Examples

$\psi = \text{round}; \text{ring graph with } n = 10 \text{ nodes.}$

$\eta = 0.1, \text{last 100 steps}$
Examples

$\psi = \text{round}; \text{ring graph with } n = 10 \text{ nodes.}$

$\eta = 0.05, \text{overall behavior}$

States

Disagreement

Drift

log$(X^T Lx)$

d(k)

time steps
Examples

ψ = round; ring graph with $n = 10$ nodes.

$\eta = 0.05$, last 100 steps

States

Disagreement

Drift

$log(x^T L x)$

$d(k)$

time steps
Conclusions

- Consensus can be reached with arbitrary precision regardless of quantization and noise.

- Possible improvements:
  - Characterization of convergence speed / precision tradeoffs with choosing $\eta$.
  - Find better bounds
    - In practice, the error appears independent of the number of nodes. However, $\lambda_n\{L\} / \lambda_2\{L\} \simeq O(n^2)$, for ring graphs.
    - Consider with specific quantization functions $\psi$ or topologies.
  - Prove that, if $\psi$ deterministic, it converges to a periodic orbit