Calibration by correlation using metric embedding from non-metric similarities

Appendix A: Proof of the main results

A-1 Definitions

For convenience of the reader, we also repeat various definitions which have been omitted or only stated informally. Introductory references for the differential geometric concepts used in this paper are [1, 2].

$\mathbb{R}_0^+$ is the set of positive reals. The generic m-sphere is $\mathbb{S}^m$. $\mathbb{S}^2$ is the 3D sphere, and $\mathbb{S}^1$ is the circle. The target manifold is $\mathbb{M}$ and d indicates its geodesic distance. The set of unknown points to be reconstructed is $\mathcal{S} = \{s_i\}$.

Definition A-1. The *radius* of a set $\mathcal{S}=\{s_i\}$ is defined as

$$\mathsf{rad}(\mathcal{S}) \triangleq \min_i \max_j \ d(s_i,s_j). \tag{1}$$

The *diameter* is twice the radius.

Definition A-2. The *informative radius* $\mathsf{infr}(f)$ of f is the maximum r such that f is invertible in $[0, r]$.

Definition A-3. An *isometry* is a map $\varphi : \mathbb{M} \to \mathbb{M}$ that preserves distances: $d(\varphi(s_i), \varphi(s_j)) = d(s_i, s_j)$.

Definition A-4. A *conformal map* is a map that preserves the angles between geodesics.

A map is conformal if and only if its Jacobian is proportional to an orthogonal matrix.

Definition A-5. A *generic warping* is a map $\varphi_m : \mathbb{M} \to \mathbb{M}$ such that $d(\varphi(s_1), \varphi(s_2)) = m(d(s_1, s_2))$, for some monotonic function $m : \mathbb{R}_0^+ \to \mathbb{R}_0^+$.

Definition A-6. A *linear warping* is a map φ_α from $\mathbb{M}$ to itself such that $d(\varphi_\alpha(s_1), \varphi_\alpha(s_2)) = \alpha d(s_1, s_2)$ for some $\alpha > 0$.

Definition A-7. A *wiggling* of a set $\{s_i\} \subset \mathbb{M}$ is a map $\varphi : \mathbb{M} \to \mathbb{M}$ that preserves the order of distances: for all i, j, k, l: $d(s_i, s_j) < d(s_k, s_l) \Leftrightarrow d(\varphi(s_i), \varphi(s_j)) < d(\varphi(s_k), \varphi(s_l))$.

Definition A-8. A *geodesic curve* $g(A, B, t)$ from point A to point B, for $t \in [0, 1]$, is the curve on the manifold such that

$$\begin{aligned} d(g(A, B, t), A) &= td(A, B), \\ d(g(A, B, t), B) &= (1 - t)d(A, B). \end{aligned}$$

In particular, $g(A, B, 0) = A$ and $g(A, B, 1) = B$.

A-2 Proof of Proposition 2

We use some language (Haar measure) from group theory; for reference, see, e.g., [3, 4]. The luminance at pixel $s \in \mathbb{S}^2$ at time t can be written as

$$y(s, t) = h(\boldsymbol{p}(t), \mathbf{R}(t)\, s),$$

where $\boldsymbol{p} \in \mathbb{R}^m$ is the sensor position, $\mathbf{R} \in \mathsf{SO}(3)$ is the sensor orientation, and $h : \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ is a function that describes the environment. In the following, we drop the dependence on time.

Proof: Consider two pairs of pixels (s_i, s_j) and (s_k, s_l) having the same distance:

$$d(s_i, s_j) = d(s_k, s_l).$$

We will show that this constraint is enough for the pairwise statistics to be equal:

$$\mathbb{E}\{g(y(s_i), y(s_j))\} = \mathbb{E}\{g(y(s_k), y(s_l))\}. \tag{2}$$

Because there is no other relation between the two pairs of pixels other than their distance, the statistics g depends only on the distance.

If the probability distribution of $\mathbf{R}$ is uniform on SO(3), that is, it is the Haar measure SO(3), then it is also invariant to a rotation (i.e., left/right actions): for all functions z and rotations $\mathbf{X}$, $\mathbb{E}\{z(\mathbf{R})\} = \mathbb{E}\{z(\mathbf{RX})\}$.

In our case, we have that for any $\mathbf{X}$,

$$\begin{aligned}\mathbb{E}\{g(y(s_i), y(s_j))\} &= \mathbb{E}\{g(h(\boldsymbol{p}, \mathbf{R}\, s_i), h(\boldsymbol{p}, \mathbf{R}\, s_j))\} \\ &= \mathbb{E}\{g(h(\boldsymbol{p}, \mathbf{R}\, \mathbf{X} s_i), h(\boldsymbol{p}, \mathbf{R}\, \mathbf{X} s_j))\} \qquad (3)\end{aligned}$$

Because $d(s_i, s_j) = d(s_k, s_l)$, there exists an $\mathbf{X}$ such that

$$s_k = \mathbf{X} s_i, \qquad s_l = \mathbf{X} s_j.$$

By substituting this $\mathbf{X}$ in (3) we obtain (2). □

A-3 Proof of Proposition 8

A-3.1 Proof overview

The starting point is considering that the largest unobservable transformations are the set of wigglings (Definition 7), because they are exactly those that keep constant the order of the inter-points distances, which is the sufficient statistics for the estimation problem. All other symmetries—isometries (Definition A-3), linear warping (Definition 6), generic warping (Definition A-5) are a specialized version of wigglings. Moreover, an isometry is a linear warping with $\alpha = 1$, and a linear warping is a specialization of a generic warping. In summary, just by the definition of the various transformations, we have the following chain of inclusions:

$$\text{isometries} \subset \begin{array}{c}\text{linear}\\ \text{warpings}\end{array} \subset \begin{array}{c}\text{generic}\\ \text{warpings}\end{array} \subset \text{wigglings}.$$

Isometries and warpings are very structured transformations, but wigglings are in general discontinuous. The next step in the analysis is understanding in what cases the set of wigglings is more structured. Proposition A-9 shows that, as the number of points becomes large (in the limit, infinite), wigglings are constrained to be generic warpings. Thus, if $\mathcal{S}$ has an infinite number of points, we have the following:

$$\text{isometries} \subset \begin{array}{c}\text{linear}\\ \text{warpings}\end{array} \subset \begin{array}{c}\text{generic}\\ \text{warpings}\end{array} \overset{n\to\infty}{=} \text{wigglings}.$$

With this assumption, we now can study a much more well-behaved set of transformations. Proposition A-10 gives the unexpected result that, in general, there exist no generic nonlinear warpings (Definition A-5), a result that does not depend on the manifold yet (i.e., we did not consider topology or curvature). Intuitively, there is no way to deform the distances in a nonlinear way that maintains the consistency of all constraints. The proof is based on an elementary argument based on the fact that any generic warping must preserve geodesics (Lemma A-11). Thus, only by assuming that the number of points is large, and with no assumptions on the manifold, we can conclude that:

$$\text{isometries} \subset \begin{array}{c}\text{linear}\\ \text{warpings}\end{array} = \begin{array}{c}\text{generic}\\ \text{warpings}\end{array} = \text{wigglings}$$

This means that the largest group of symmetries of the problem is composed by linear warping. At this point, we have to consider the property of the manifold. Proposition A-12 shows that, if the manifold has nonpositive curvature, then all linear warpings are necessarily isometries (the scaling factor is 1):

$$\text{isometries} \overset{\begin{array}{c}\mathbb{M}\\ \text{curved}\end{array}}{=} \begin{array}{c}\text{linear}\\ \text{warpings}\end{array} = \begin{array}{c}\text{generic}\\ \text{warpings}\end{array} = \text{wigglings}$$

This means that for the sphere $\mathbb{S}^2$ and the hyperbolic plane, isometries are the largest group of symmetries. This is surprising, because it means that *we can recover the scale*, even though the measurements available are completely non-metric. Instead, for Euclidean spaces, it is easy to see that a linear warping is always unobservable. Finally, Proposition A-13 discusses the special case of the circle. Because the topology is not simply connected, it is possible to establish additional constraints: intuitively, an arbitrary warping is not allowed, because if the distribution $\mathcal{S}$ is inflated too much, the tails will "crash" into each other and violate the problem constraints.

A-3.2 Proof details

Proposition A-9. *If $\mathcal{S}$ is a connected open set, all wigglings are generic warpings.*

Proof: The intuition is that a non-trivial wiggling is possible only if there are "gaps" between the points; as the points get denser, the gaps close and the wiggling degenerates to a warping.

Note that the definition of wiggling does not imply any particular property of the map φ such as continuity. It is a map defined only the subset $\mathcal{S}$ of $\mathbb{M}$. There is no information of how φ behaves outside of $\mathcal{S}$. However, if $\mathcal{S}$ is an open subset of $\mathbb{M}$, then necessarily φ must have certain regularities.

First of all, it should necessarily be a continuous map. This can be seen directly from the relation $d(s_i, s_j) < d(s_k, s_l) \Leftrightarrow d(\varphi(s_i), \varphi(s_j)) < d(\varphi(s_k), \varphi(s_l))$ if we let $s_i = s_k$ and consider two sequences $s_j^{(m)} \overset{m\to\infty}{\to} s_i$ and $s_l^{(m)} \overset{m\to\infty}{\to} s_k$.

Consider two pairs of points s_i, s_j at distance $\delta = d(s_i, s_j)$. Consider two other pairs of points s_k, s_l with the same relative distance $\delta = d(s_k, s_l)$ — because the set is open, and the distance is continuous, s_k can be found in a neighborhood of s_i and s_j in a neighborhood of s_l. Because $d(s_i, s_j) = d(s_k, s_l)$, the wiggling direction constraint implies that $d(\varphi(s_i), \varphi(s_j)) = d(\varphi(s_k), \varphi(s_l))$. Because s_k, s_l have no other relation to s_i, s_j other than their distance, it follows that the distance of two points transformed by φ only depends on their initial distance: $d(\varphi(s_i), \varphi(s_j)) = m(d(s_i, s_j))$, for some possibly nonlinear function m. Because φ is continuous, this holds for all points in $\mathcal{S}$, therefore φ is a generic warping. □

Proposition A-10. *All generic warpings are linear warpings.*

Proof: The proof relies on Lemma A-11 below, which says that generic warpings preserve the geodesics. This means that, if the midpoint between A and B is C, then $\varphi(C)$ is the midpoint between $\varphi(A)$ and $\varphi(C)$. Let $d(A, C) = d(C, B) = \ell$. Then $d(\varphi(A), \varphi(C)) = d(\varphi(C), \varphi(B)) = m(\ell)$. We can find two different expressions for $d(\varphi(A), \varphi(B))$:

$$d(\varphi(A), \varphi(B)) = m(d(A, B)) = m(2\ell), \quad \text{and}$$
$$d(\varphi(A), \varphi(B)) = d(\varphi(A), \varphi(C)) + d(\varphi(C), \varphi(B)) = 2m(\ell).$$

It follows that $m(\ell) = \frac{1}{2}m(2\ell)$. Generalize this reasoning to an equal division of the geodesics in k parts, to derive $m(x) = \frac{1}{k}m(kx)$, for all $x > 0$ and integers $k \geq 1$. Take the derivative of both sides with respect to x to obtain $m'(x) = m'(kx)$. For any $y > 0$, let $x = y/k > 0$, and let $k \to \infty$, to obtain $m'(y) = m'(0)$, which implies that m is a linear function. □

Lemma A-11. *A generic warping preserves geodesics. More formally, for $A, B \in M$ and $t \in [0, 1]$, let $g(A, B, t)$ be the geodesic between A and B. If $\varphi : M \to M$ is a warping, then $g(\varphi(A), \varphi(B), t) = \varphi(g(A, B, t))$.*

Proof: As a base case, we prove the statement for the midpoint. Suppose that there exists a geodesic between A and B. Let C be the midpoint between A and B, with $d(A, C) = d(C, B) = L$. Let $a = \varphi(A)$ and $b = \varphi(B)$ be the transformed points. Let $c = g(a, b, \frac{1}{2})$ be the midpoint between a and b, with $d(a, c) = d(c, b) = \ell$. Using some elementary properties of geodesics, we shall derive that $\varphi(C) = c$.

Because c is the midpoint, the shortest path between a and b goes through c:

$$d(a, c) + d(c, b) \leq d(a, x) + d(x, b), \qquad \text{for all } x.$$

Write this for $x = \varphi(C)$:

$$d(a, c) + d(c, b) \leq d(a, \varphi(C)) + d(\varphi(C), b)$$

On the right-hand side, substitute $d(a, \varphi(C)) = d(\varphi(A), \varphi(C)) = m(d(A, C))$, using the definition of warping. Likewise $d(\varphi(C), b) = d(\varphi(C), \varphi(B)) = m(d(C, B))$, giving

$$d(a, c) + d(c, b) \leq m(d(A, C)) + m(d(C, B)).$$

The point c is the midpoint, so let $\ell = d(a, c) = d(c, b)$, and $L = d(A, C) = d(C, B)$. We obtain that $\ell \leq m(L)$.

We can do the same computation with A and B. Because C is the midpoint between A and B, we have that $d(A, C) + d(C, B) \leq d(A, x) + d(x, B)$, for all x. Write it for $x = \varphi^{-1}(c)$ and substitute $A = \varphi^{-1}(a)$ and $B = \varphi^{-1}(b)$ to obtain $2L \leq d(A, \varphi^{-1}(c)) + d(\varphi^{-1}(c), B) = d(\varphi^{-1}(a), \varphi^{-1}(c)) + d(\varphi^{-1}(c), \varphi^{-1}(b)) = m^{-1}(d(a, c)) + m^{-1}(d(c, b)) = 2m^{-1}(\ell)$, which gives us $\ell \geq m(L)$. Together with $\ell \leq m(L)$, we conclude that $\ell = m(L)$. This means that $d(a, \varphi(C)) = d(\varphi(C), b) = \ell$, and hence $\varphi(C)$ is the midpoint between a and b. Because the midpoint is unique, it follows that $c = \varphi(C)$.

We have proved that $g(\varphi(A), \varphi(B), \frac{1}{2}) = \varphi(g(A, B, \frac{1}{2}))$. By dividing the original geodesics, and applying the reasoning above recursively, one can show that $g(\varphi(A), \varphi(B), \frac{a}{2^b}) = \varphi(g(A, B, \frac{a}{2^b}))$ for all integers $b \geq 0$ and $a \leq 2^b$. The set of dyadic rationals $a/2^b$ is dense in $[0, 1]$, and the functions $t \mapsto g(\varphi(A), \varphi(B), t)$ and $t \mapsto \varphi(g(A, B, t))$

are continuous, because they are compositions of continuous functions. If two continuous functions on the same domain X agree on a dense subset of X, the agree on the whole domain. Therefore, it holds that $g(\varphi(A), \varphi(B), t) = \varphi(g(A, B, t))$ for all $t \in [0, 1]$. □

Proposition A-12. *For $\mathbb{S}^m, m \geq 2$ and the hyperbolic plane, all linear warpings are isometries.*

Proof: This is true for all manifolds with nonzero curvature, but the m-sphere and the hyperbolic plane admit an elementary proof based on spherical/hyperbolic geometry. Firstly, note that a linear warping is a conformal map (Definition A-4) as the Jacobian is uniformly α times an orthogonal matrix. Conformal maps preserve angles between geodesics.

Now it is time to recall high-school facts about spherical geometry: the sides of a spherical triangle are uniquely determined by its angles. The same is true for the hyperbolic plane [2].

Consider now three points in $\mathcal{S}$ and the induced spherical/hyperbolic triangle. Under a linear warping, its internal angles are preserved because a linear warping is conformal. Because the angles are preserved, the sides of the triangle are preserved as well, and therefore the distance between points is unchanged. Hence any linear warping is an isometry. □

Proposition A-13. *If $\mathbb{M} = \mathbb{S}^1$ and $\mathsf{rad}(\mathcal{S}) + \mathsf{infr}(f) < 2\pi$, a linear warping with $\alpha \leq (2\pi - \mathsf{rad}(\mathcal{S}))/\mathsf{infr}(f)$ is unobservable.*

Proof: (sketch) This can be verified directly; the upper bound on α ensures that the tails of $\mathcal{S}$ do not overlap in the informative range of f. This result does not hold for $\mathbb{S}^2$, where the geometry of the problem constrains linear warpings to be isometries ($\alpha = 1$). □

References

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